GCSE Mathematics: Proofs

Priority Learning

Worksheet

Name: ______________________________ **Date**: ______________

Aims of this revision sheet:

- Learning how to start off proofs.
- Being able to recognise different types of proof.

How To Start

In this section we learn how to start proof questions.

Odd and Even Integers

If our proofs involve an even or odd integer we start them as follows.

- Let n be an integer.
- Any number multiplied by 2 is even.
- Any even number $(2n) + 1$ or -1 is odd.

Even	Odd
$2n$	$2n + 1$

Some proofs ask for consecutive numbers, the table below shows what you should do in each case.

- Note that to get from an odd number to the consecutive odd number we have to add 2. Adding 1 would just take us to the next number which would be even.
- The same logic applies for getting from an even number to the consecutive even number. We need to add 2.

Any Consecutive Numbers	Even	Odd
$n - 1$	$2n - 2$	$2n - 1$
n	$2n$	$2n + 1$
$n + 1$	$2n + 2$	$2n + 3$

Some questions just want any two numbers, they don't have to be consecutive.

- It makes sense to choose x and y, these have no relation to each other.

Any Numbers	Even	Odd
x	$2x$	$2x + 1$
y	$2y$	$2y + 1$

Square Numbers

Some proofs require us to square numbers and prove various things.

- Let n be an integer.

Any Number Squared	Even Number Squared	Odd Number Squared
n	$(2n)^2 = 4n^2$	$(2n+1)^2 = 4n^2 + 2n + 2n + 1$ $= 4\text{n}^2 + 4n + 1$

How To Conclude Proofs

In this section we learn how to conclude proof questions.

Proving Something Is Odd/Even

We conclude proofs in much the same way we start them.

- To prove something is even we want to show it is a multiple of 2. So factor a 2 out of the algebraic expression we obtain.
- To prove something is odd we want to show it is a multiple of 2, with 1 added on. So factor a 2 out of the algebraic expression we obtain, then we will see that there is a remainder of 1.

Even (examples)	Odd (examples)
$4n^2 - 2n + 10$ $= 2(2n^2 - n + 5)$	$6n^2 - 4n + 1$ $= 2(3n^2 - 2n) + 1$ Even - odd = odd.
$10n^3 - 4$ $= 2(5n^3 - 2)$	$2n^2 + 8n - 5$ $= 2(n^2 + 4n) - 5$ Even - odd = odd.

We use the same logic when we aim to prove something is a multiple of another number, as we will see in the example below.

Example - Proving Divisions/Multiples

Prove that $(n+4)^2-(n+2)^2$ is always divisible by 4 for all positive integer values of n.

Solution: More often than not, when we have brackets and are asked to prove something we need to expand the brackets and then simplify.

$$\begin{aligned}(n+4)^2-(n+2)^2 &= n^2+4n+4n+16-(n^2+2n+2n+4) \\ &= n^2+4n+4n+16-n^2-4n-4 \qquad = 4n+12 \\ &= 4(n+3)\end{aligned}$$

When we are trying to show an expression is divisible by something that is the same as showing it is a multiple of something. The method is to factor that number out of your expression. In this example, $4(n+3)$ is divisible by 4, because it is a multiple of 4.

Example

Show that $(n+2)^2-(n-2)^2+3$ is always odd for all positive integer values of n.

Solution:

$$\begin{aligned}(n+2)^2-(n-2)^2+3 &= n^2+2n+2n+4-(n^2-2n-2n+4)+3 \\ &= n^2+2n+2n+4-n^2+4n-4+3 \\ &= 8n+3 \\ &= 2(4n)+3\end{aligned}$$

$2(4n)$ is even, 3 is odd. Even + odd = odd.

Example

Prove that the sum of any three consecutive integers is divisible by 3.

Solution: 3 consecutive integers are $n, n+1$ and $n+2$. To find their sum we add them up.

$$n+n+1+n+2=3n+3$$

To show it is divisible by 3 we factor a 3 out. So we get

$$3(n+1)$$

Which is a multiple of 3. Hence, it is divisible by 3.

Example

Prove that the product of two consecutive odd numbers is also odd.

Solution: Let n be an integer. Then $2n-1$ and $2n+1$ are consecutive odd integers. To find the product we multiply them.

$$\begin{aligned}(2n-1)(2n+1) &= 4n^2 + 2n - 2n - 1 \\ &= 4n^2 - 1 \\ &= 2(2n^2) - 1\end{aligned}$$

$2(2n^2)$ is even, -1 is odd. Even - odd = odd.

Example

Prove that the sum of the squares of two consecutive even numbers is also even.

Solution: Let n be an integer. Then $2n$ and $2n+2$ are consecutive even integers. To find the sum of the squares first we square them then we add them (BIDMAS dicates this order).

$$\begin{aligned}(2n)^2 + (2n+2)^2 &= 4n^2 + 4n^2 + 4n + 4n + 4 \\ &= 8n^2 + 8n + 4 \\ &= 2(4n^2 + 4n + 2)\end{aligned}$$

Which is even.